Hale School

Mathematics Specialist

Test 1 --- Term 1 2019

Complex Numbers

/ 46

Name: SOLUTIONS

Instructions:

- **Calculators are NOT allowed**
- **External notes are not allowed**
- **Formula Sheet will be provided**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

All arguments must be given using principal values.

1. **[2, 3 and 3 = 8 marks]**

(a) Find

i) $\mathrm{Re}\left(\frac{2+3i}{1-i}\right) = Re\left(\frac{(2+3i)(1+i)}{2}\right)$ ✓ uses conjugate to simplify correctly

$= \frac{2-3}{2}$

$= -\frac{1}{2}$ ✓ correct answer

ii) $\mathrm{Im}\left(2cis\left(\pi/3\right)+3cis\left(\pi/4\right)\right)$

$= Im\left(1+\sqrt{3}i + \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right)$ ✓ $1+\sqrt{3}i$

✓ $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$= \sqrt{3} + \frac{3\sqrt{2}}{2}$ ✓ answer

(b) Simplify

$$\frac{\left(\sqrt{2}cis\left(3\pi/4\right)\right)^5}{\left(2cis\left(\pi/6\right)\right)^2}$$ leaving your answer in polar form, $rcis\theta$.

$= \frac{(\sqrt{2})^5}{2^2} cis\left(\frac{3\pi}{4}\times 5 - \frac{2\pi}{6}\right)$ ✓ correct use of De Moivre's Theorem

$= \sqrt{2}\, cis\left(\frac{41\pi}{12}\right)$ ✓ finds $\sqrt{2}$ modulus

$= \sqrt{2}\, cis\left(-\frac{7\pi}{12}\right)$ ✓ finds $-\frac{7\pi}{12}$ argument

2. **[1, 1, 1 and 2 = 5 marks]**

In the Argand plane below, a unit circle has been drawn and P is the point corresponding to the complex number z.

In the diagram, clearly mark the complex numbers corresponding to:

i) z^2

ii) $\frac{1}{z}$

iii) $-2z$

iv) $\bar{z} - z$ $= (x - yi) - (x + yi)$

$= -2yi$

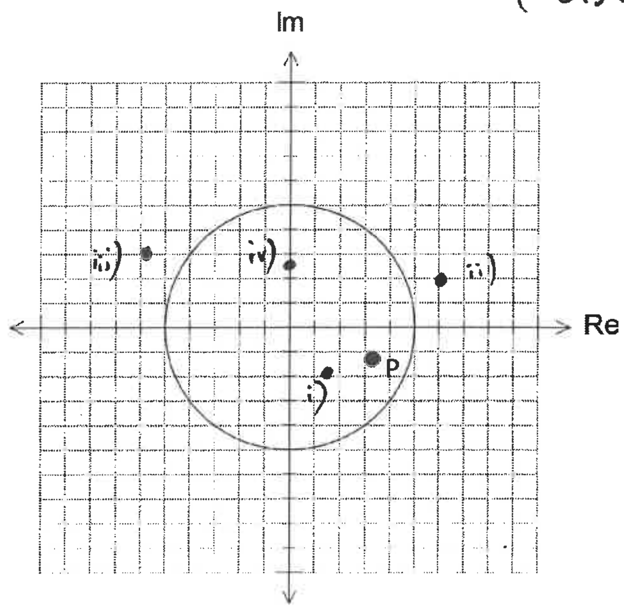
✓ approx location for z^2
(double angle, smaller modulus)

✓ approx location for $\frac{1}{z}$
(opposite angle, outside circle)

✓ approx location for $-2z$
(opposite direction, twice modulus)

✓ $\bar{z} - z$ is purely imaginary

✓ approx location for $\bar{z} - z$
(0.5i)



3. **[4 marks]**

Let $z = a + bi$ be any complex number.

Show that the locus of points for which $\operatorname{Im}\left(\frac{z-2}{z}\right) = 1$ is a circle.

$$\operatorname{Im}\left(\frac{a+bi-2}{a+bi}\right) = 1$$

$$\Rightarrow \operatorname{Im}\left(\frac{(a-2+bi)(a-bi)}{a^2+b^2}\right) = 1$$ ✓ uses conjugate

$$\Rightarrow \frac{ab-(a-2)b}{a^2+b^2} = 1$$ ✓ finds the imaginary part

$$\Rightarrow \frac{2b}{a^2+b^2} = 1$$ ✓ rearranges correctly

$$\Rightarrow a^2+b^2 = 2b$$ ✓ justifies circle

$$\Rightarrow a^2 + (b-1)^2 = 1$$

which is a circle centre $(0,1)$ radius 1

4. **[3 and 3 = 6 marks]**

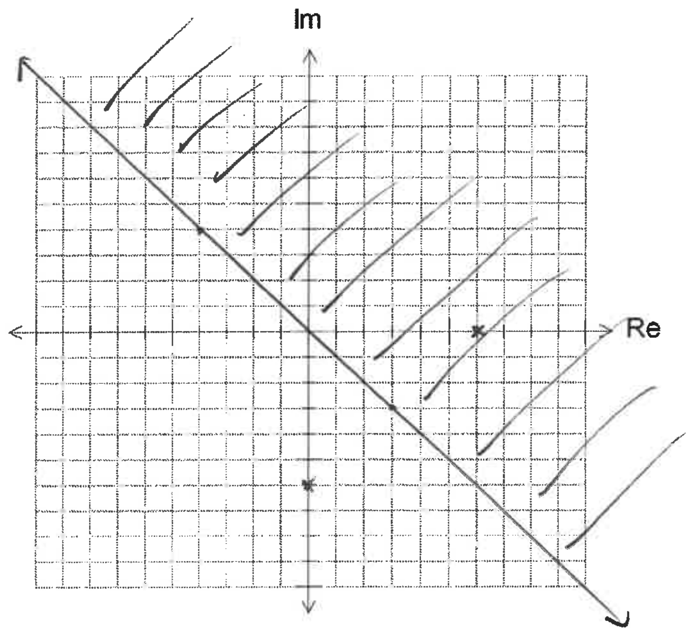
Sketch the following loci on the complex planes provided.

i) $|z-3| \le |z+3i|$

✓ locates 3 and $-3i$

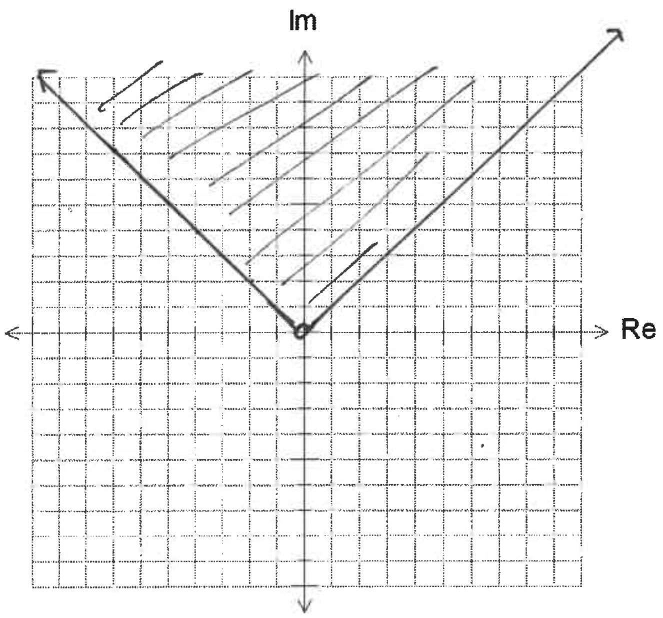
✓ draws perpendicular bisector

✓ shading correct side



ii) $\arg\left((1+i)z\right) \ge \frac{\pi}{2}$

After rotation of $\pi/2$ end up in 2^{nd} quadrant



✓ identifies rotation of $\pi/4$ (States or uses $\arg z = \pi/4$)

✓ uses $\arg z = 3\pi/4$

✓ shading and open circle at $(0,0)$

5. **[5 marks]**

Write down in *cis* (polar) form the solutions to the equation $z^5 = \frac{1}{64}\left(-1+\sqrt{3}\,i\right)$.

$$z^5 = \frac{1}{32}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$z^5 = \left(\frac{1}{2}\right)^5 \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

$$z_1 = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15}\right)$$

$$z_2 = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15} + \frac{2\pi}{5}\right) = \frac{1}{2}\operatorname{cis}\left(\frac{8\pi}{15}\right)$$

$$z_3 = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15} + \frac{4\pi}{5}\right) = \frac{1}{2}\operatorname{cis}\left(\frac{14\pi}{15}\right)$$

$$z_4 = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15} - \frac{2\pi}{5}\right) = \frac{1}{2}\operatorname{cis}\left(-\frac{4\pi}{15}\right)$$

$$z_5 = \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15} - \frac{4\pi}{5}\right) = \frac{1}{2}\operatorname{cis}\left(\frac{-10\pi}{15}\right) = \frac{1}{2}\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

Solutions are

$$\frac{1}{2}\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ \frac{1}{2}\operatorname{cis}\left(-\frac{4\pi}{15}\right),\ \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{15}\right),\ \frac{1}{2}\operatorname{cis}\left(\frac{8\pi}{15}\right),\ \frac{1}{2}\operatorname{cis}\left(\frac{14\pi}{15}\right)$$

✓ expresses z^5 in cis form

✓ finds principal solution

✓ adds $\frac{2\pi}{5}$ to find other solutions

✓ identifies all solutions

✓ gives answers in simplified form with correct domain

6. **[4, 4 = 8 marks]**

(a) When the polynomial $z^2+(2-i)z+Ai+B$ is divide by $z+2i$ the remainder is $2+4i$. Determine the values of A and B.

$f(-2i) = 2+4i$ ✓ subs 2i

$\therefore (-2i)^2 + (2-i).(-2i) + Ai + B = 2+4i$ ✓ evaluation

$\therefore -4 - 4i - 2 + Ai + B = 2+4i$ ✓ A=8

$A = 8$ ✓ B=8

$B = 8$

(b) Consider the polynomial $Q(z) = z^4 - 6z^3 + 5z^2 + 22z + 38$. Given that one solution to the equation $Q(z) = 0$ is $z = 4-\sqrt{3}\,i$, find the other solutions.

$z = 4-\sqrt{3}i$ and $z = 4+\sqrt{3}i$ both solutions ✓ 2nd sol^n

$\Rightarrow (z-4+\sqrt{3}i)(z-4-\sqrt{3}i)$ a factor ✓ multiplies factors

$\Rightarrow (z-4)^2+3$ a factor

$\Rightarrow z^2-8z+19$ a factor

✓ finds z^2+2z+2

$Q(z) = (z^2-8z+19)(z^2+2z+2)$

$\therefore$ Other solutions are $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$z = 4+\sqrt{3}i$ ✓ gives 3 other soltions

$z = -1+i$

$z = -1-i$

7. **[2, 2 and 5 = 9 marks]**

a) Given that $z = cis\theta$,

i) prove that $z - \frac{1}{z} = 2i\sin(\theta)$

$$
\begin{aligned}
LHS &= z - \frac{1}{z} \\
&= cis\theta - \frac{cis 0}{cis\theta} \\
&= \cos\theta + i\sin\theta - cis(-\theta) \\
&= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta)) \\
&= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) \\
&= 2i\sin\theta \\
&= RHS
\end{aligned}
$$

✓ uses $cis(-\theta)$

✓ full and correct proof

ii) use de Moivre's Theorem to prove that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$

$$
\begin{aligned}
LHS &= z^n + \frac{1}{z^n} \\
&= (cis\theta)^n + (cis\theta)^{-n} \\
&= cis(n\theta) + cis(-n\theta) \\
&= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\
&= \cos n\theta + i\sin n\theta + \cos(n\theta) - i\sin(n\theta) \\
&= 2\cos(n\theta) \\
&= RHS
\end{aligned}
$$

✓ uses De M correctly

✓ full and correct proof

7 Continued

(b) Use the results from part (a) to show that

$\sin^4\theta - \sin^2\theta = a\cos(4\theta) - b$, giving the values of a and b.

$$\sin^4\theta = \left(\frac{1}{2i}\right)^4\left(z - \frac{1}{z}\right)^4 = \frac{1}{16}\left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)$$

$$\sin^2\theta = \left(\frac{1}{2i}\right)^2\left(z - \frac{1}{z}\right)^2 = -\frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)$$

$$\therefore \quad \sin^4\theta - \sin^2\theta = \frac{1}{16}\left(z^4 + \frac{1}{z^4}\right) - \frac{1}{4}\left(z^2 + \frac{1}{z^2}\right) + \frac{3}{8} + \frac{1}{4}\left(z^2 - 2 + \frac{1}{z^2}\right)$$

$$= \frac{1}{16}\left(2\cos 4\theta\right) + \frac{3}{8} - \frac{1}{2}$$

$$= \frac{1}{8}\cos 4\theta - \frac{1}{8}$$

$$\therefore \quad \underline{a = \frac{1}{8}}, \quad \underline{b = \frac{1}{8}}$$

✓ expands $\left(z - \frac{1}{z}\right)^4$

✓ expands $\left(z - \frac{1}{z}\right)^2$

✓ substitutes to find $\sin^4\theta - \sin^2\theta$

✓ justifies result and finds a, b